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A formulation is given and a procedure is proposed for constructing a confidence interval for a certain ordered (location or scale) parameter and for simultaneously selecting all populations having parameters equal or larger than this ordered parameter with a preassigned minimal probability. The well-known indifference-zone formulation of the ranking problem is obtained as a special case as is the problem of interval estimation of an ordered parameter.

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ON INTERVAL ESTIMATION AND SIMULTANEOUS
SELECTION OF ORDERED LOCATION OR SCALE PARAMETERS

by

M. Haseeb Rizvi and K. M. Lal Saxena

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ON INTERVAL ESTIMATION AND SIMULTANEOUS SELECTION OF ORDERED LOCATION OR SCALE PARAMETERS

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1. Introduction and Formulation of the Problem .

Procedures for selection of a certain number of populations with larger parameters from a collection of several populations have been studied extensively in the past two decades; see, for example, Barr and Rizvi [1] for a simple exposition. Recently Saxena and Tong [2] and Saxena [3] have considered confidence intervals for the largest parameter. The present paper attempts to combine these two requirements simultaneously in a single formulation. The problem of interest is to construct a confidence interval for a certain ordered parameter and simultaneously select all populations having parameters equal or larger than this ordered parameter, with a preassigned minimal probability whenever parameters lie in a specified subspace. A procedure R is proposed to solve this problem, and its performance in terms of probability requirement being satisfied is evaluated.

Consider $k(\geq 1)$ populations $\pi_i (i=1,\ldots,k)$ with absolutely continuous distribution function (df) $F(.; \theta_i)$ of Y_i on the real line with real parameter θ_i and let $f(.; \theta_i)$ be the corresponding density. Let $\theta_{[1]} \leq \theta_{[2]} \leq \cdots \leq \theta_{[k]}$ denote the ordered values of

the components of $\underset{\sim}{\theta} = (\theta_1, \theta_2, \ldots, \theta_k) \in \Omega$. For $1 \le t \le k$, we require a procedure R that selects all π_i with $\theta_i \ge \theta_{[k-t+1]} = \theta$ (say) and simultaneously gives an interval I such that $\theta \in I$. Denote by CS the (correct) selection of all π_i with $\theta_{[i]}, i = k-t+1, \ldots, k$ and by CD (correct decision) the inclusion of θ in I and let $P(\underset{\sim}{\theta})$ denote $\Pr\{CS \cap CD | R\}$. Then the procedure R, for some preassigned constant γ, $1/\binom{k}{t} < \gamma < 1$, is more specifically required to satisfy

$$\inf_{\Omega(\psi)} P(\underset{\sim}{\theta}) \ge \gamma , \tag{1.1}$$

where $\Omega(\psi) = \{\underset{\sim}{\theta} \in \Omega : \theta_{[k-t]} \le \psi(\theta_{[k-t+1]})\}$ and ψ is a given function on the real line such that $\psi(x) \le x$.

2. Main Results on $P(\underset{\sim}{\theta})$ for Proposed R.

Proposed Procedure R.

Rank $Y_1, Y_2, \ldots, Y_k$, breaking ties (if any) with suitable randomization, and let $Y_{[i]}$ be the i^{th} smallest Y_i. Consider two suitably chosen continuous increasing functions h_1 and h_2 (with inverses g_2 and g_1 respectively). Construct the random interval $I_0 = (h_1(Y_{[k-t+1]}), h_2(Y_{[k-t+1]}))$. Then assert that $\theta \in I_0$ and that the π_j's corresponding to $Y_{[j]} (j = k-t+1, \ldots, k)$ have parameters $\theta_j \ge \theta$.

We shall presently investigate the infimum of $P(\underset{\sim}{\theta})$ over $\Omega(\psi)$ for the above R and later determine conditions so that R satisfies (1.1). We have

$$(2.1) \qquad P(\underset{\sim}{\theta}) = \sum_{j=k-t+1}^{k} \int_{g_1(\theta)}^{g_2(\theta)} \prod_{r=1}^{k-t} F(y;\ \theta_{[r]}) \prod_{\substack{s=k-t+1 \\ s \neq j}}^{k} (1 - F(y;\ \theta_{[s]}))$$

$$dF(y;\ \theta_{[j]}) \ .$$

An obvious proposition follows.

Proposition 1.

A sufficient condition that $P(\underset{\sim}{\theta})$ be a nonincreasing function of $\theta_{[1]}, \ldots, \theta_{[k-t]}$ is that the df's $F(\cdot,\ \theta_i)$, $i = 1, \ldots, k$ be stochastically ordered.

Location Parameter Case.

Let $F(y,\ \theta_i) = F(y - \theta_i)$, $\psi(\theta) = \theta - \delta$, $g_1(\theta) = \theta - a$, $g_2(\theta) = \theta + b$, where $\delta \geq 0$ and a and b with $a + b > 0$ are given constants; $\Omega(\psi)$ will now be denoted by $\Omega(\delta)$. Clearly, (2.1) implies

Proposition 2.

For $t = 1$,

$$(2.2) \qquad \inf_{\Omega(\delta)} P(\underset{\sim}{\theta}) = \int_{-a}^{b} F^{k-1}(y+\delta)\,dF(y) \ .$$

Theorem 1.

Suppose $f(y-\theta_1)$ has a monotone likelihood ratio (m.l.r.) in y for θ_1 and constants a and b are chosen such that $a+b > 0$ and

$$(2.3) \qquad F(-a) + F(b) \geq 1 \quad .$$

Then, for $1 < t \leq k$,

$$(2.4) \qquad \inf_{\Omega(\delta)} P(\underset{\sim}{\theta}) = P(\underset{\sim}{\theta}_o) = t \int_{-a}^{b} F^{k-t}(y+\delta)[1-F(y)]^{t-1} dF(y),$$

where $\underset{\sim}{\theta}_o$ has first $(k-t)$ components equal to $(\theta-\delta)$ and the last t components equal to θ, θ being any arbitrary value of $\theta_{[k-t+1]}$.

Proof.

Since $f(y, \theta_1)$ has an m.l.r., Proposition 1 implies that $P(\underset{\sim}{\theta})$ is minimized over Ω_1 by setting $\theta_{[1]} = \ldots = \theta_{[k-t]} = \theta-\delta$, where Ω_1 is the subset of $\Omega(\delta)$ for which $\theta_{[k-t+1]}, \ldots, \theta_{[k]}$ are held fixed. Letting $\theta-\theta_{[j]} = \delta_j (\leq 0)$, $j=k-t+2, \ldots, k$, we obtain from (2.1) after some simplification,

$$
\begin{aligned}
(2.5) \qquad \inf_{\Omega_1} P(\underset{\sim}{\theta}) &= F^{k-t}(\delta-a)[1-F(-a)] \prod_{j=k-t+2}^{k} [1-F(\delta_j-a)] \\
&+ F^{k-t}(\delta+b)[1-F(b)] \prod_{j=k-t+2}^{k} [1-F(\delta_j+b)] \\
&+ (k-t) \int_{\delta-a}^{\delta+b} F^{k-t-1}(y)[1-F(y-\delta)] \prod_{j=k-t+2}^{k} \\
&\quad [1-F(y-\delta+\delta_j)]\, dF(y) \\
&= H(\underset{\sim}{\delta}), \text{ say,}
\end{aligned}
$$

where $\underset{\sim}{\delta} = (\delta_{k-t+2},\dots,\delta_k)$. Since $H(\underset{\sim}{\delta})$ is a symmetric function of its arguments, minimization of $H(\underset{\sim}{\delta})$ over $\{\underset{\sim}{\delta}: \delta_{k-t+2} \geq \dots \geq \delta_k\}$ is equivalent to its minimization over $\Omega_2 = \{\underset{\sim}{\delta}: \delta_{k-t+2} \leq 0,\dots,\delta_k \leq 0\}$. For some j, fix $\delta_{k-t+2},\dots,\delta_{j-1},\delta_{j+1},\dots,\delta_k$ and consider $(\partial/\partial\delta_j)H(\underset{\sim}{\delta})$. Observe that the m.l.r condition implies that $f(\delta_j-a)/f(\delta_j+b)$ and $f(y+\delta_j-\delta)/f(\delta_j+b)$ are increasing functions of δ_j, for all $y \in (\delta-a, \delta+b)$. Arguing in a similar manner as in Saxena [5], we conclude that $(\partial/\partial\delta_j)H(\underset{\sim}{\delta})$ has at most one change of sign from positive to negative and consequently $\inf_{\delta_j} H(\underset{\sim}{\delta})$ is either at $\delta_j = 0$ or at $\delta_j = -\infty$. This conclusion is valid for every other j. Therefore infinum of $H(\underset{\sim}{\delta})$ over Ω_2 is achieved when a certain number r of

δ_j's are zero and the rest equal to $-\infty$; denote this infimum by $G(r)$. Then (2.5) after integration by parts gives

$$G(r) = (r+1)\int_{-a}^{b} F^{k-1}(y+\delta)[1-F(y)]^r dF(y) , \tag{2.6}$$

where $r \in \{0,1,\ldots,t-1\}$. Now it follows from the lemma given below that $G(t-1) \le G(r)$ for all $r = 0,1,\ldots,t-1$. Consequently,

$$\inf_{\Omega(\delta)} P(\underset{\sim}{\theta}) = \inf_{\Omega_2} H(\underset{\sim}{\delta}) = G(t-1), \tag{2.7}$$

which proves the theorem.

Lemma.

A sufficient condition for $G(r)$, given by (2.6), to be nonincreasing in r is that a and b are such that $F(-a)+F(b) \ge 1$.

Proof.

Consider the following density function

$$h(y;\, r) = [C(r)]^{-1}(r+1)[1-F(y)]^r f(y), -a < y < b, \quad \text{where} \tag{2.8}$$

$$C(r) = [1-F(-a)]^{r+1} - [1-F(b)]^{r+1} . \tag{2.9}$$

With E_r denoting the expectation with respect to (2.8), we can write (2.6) as

$$G(r) = C(r)E_r\{F^{k-t}(Y+\delta)\} \quad . \tag{2.10}$$

Since $h(y; r)/h(y; s)$ is an increasing function of y for $r > s$,

$$E_r\{F^{k-t}(Y+\delta)\} \geq E_s\{F^{k-t}(Y+\delta)\} \quad . \tag{2.11}$$

Therefore, $G(r) \geq G(s)$ if $C(r) \geq C(s)$ which is implied by the condition of the lemma.

Scale Parameter Case

Let $F(y; \theta_i) = F(y/\theta_i), y > 0, \theta_i > 0, \psi(\theta) = \rho\theta, g_1(\theta) = \theta/a, g_2(\theta) = \theta/b$, where ρ, a, b are given constants such that $0 < \rho \leq 1$, $0 \leq b < a$; $\Omega(\psi)$ will now be denoted by $\Omega(\rho)$. We now state the following results, the proofs for which are readily constructed along the lines of the ones given for the location parameter case.

Proposition 3.

For $t = 1$,

$$\inf_{\Omega(\rho)} P(\underset{\sim}{\theta}) = \int_{1/a}^{1/b} F^{k-1}(y/\rho)dF(y) \quad . \tag{2.12}$$

Theorem 2.

Suppose $f(y/\theta_i)$ has an m.l.r. in y for θ_i and constants a and b are chosen such that

$$(2.13) \qquad F(1/a) + F(1/b) \geq 1 .$$

Then, for $1 < t \leq k$,

$$(2.14) \qquad \inf_{\Omega(\rho)} P(\underset{\sim}{\theta}) = P(\underset{\sim}{\theta}_o) = t \int_{1/a}^{1/b} F^{k-t}(y/\rho)[1 - F(y)]^{t-1} dF(y) ,$$

where $\underset{\sim}{\theta}_o$ now has first $(k - t)$ components equal to $\rho\theta$ and the last t components equal to θ, θ being any arbitrary value of $\theta_{[k-t+1]}$.

3. Some Other Formulations as Special Cases.

A noteworthy feature of the present formulation is that the $\Pr\{CS \cap CD|R\}$ is minimized at $\underset{\sim}{\theta}_o$, defined after (2.4) in the location parameter case and after (2.14) in the scale parameter case. This $\underset{\sim}{\theta}_o$ is also the "least favorable configuration" for the indifference zone formulation of the ranking problem (see [1]) as well as for the confidence interval formulation (see [2] and [3]). Thus the present work includes the ranking formulation as a special case; with $a = b = \infty$ in the location parameter case and $a = \infty$, $b = 0$ in the scale parameter case, $\Pr\{CS \cap CD|R\}$ equals $\Pr\{CS|R\}$ and (2.2) and (2.4) reduce to (7) of [1] and (2.12) and (2.14) to (10) of [1].

The present formulation also includes the confidence interval formulation for the largest or the smallest parameter as a special case. For $t = k$ we have $\theta = \theta_{[1]}$, $\Omega(\delta) \equiv \Omega$ and $\Pr\{CS \cap CD|R\}$ equals $\Pr\{CD|R\}$.

Thus for the smallest location parameter (2.4) yields

$$\inf_{\Omega} P(\underset{\sim}{\theta}) = [1 - F(-a)]^k - [1 - F(b)]^k \,, \tag{3.1}$$

provided $F(-a) + F(b) \leq 1$. Letting $Y_i' = -Y_i$ and $\theta_i' = -\theta_i$ $(i = 1, \ldots$

we obtain for the largest location parameter,

$$\begin{aligned} P(\underset{\sim}{\theta}) &= Pr\{Y_{[k]} - b < \theta_{[k]} < Y_{[k]} + a\} \\ &= Pr\{\theta'_{[1]} - b < Y'_{[1]} < \theta'_{[1]} + a\} \end{aligned} \tag{3.2}$$

and, therefore in view of (3.1),

$$\inf_{\Omega} P(\underset{\sim}{\theta}) = F^k(b) - F^k(-a) \,, \tag{3.3}$$

provided $F(-a) + F(b) \leq 1$. Note that with $a = b = d$ and $F \equiv G_n$, (3.3) reduces to (4) of [2]. Similar discussion holds for the scale parameter case and the related result of [3].

4. Applications.

Consider k populations π_i with real parameters θ_i, $i = 1, \ldots, k$. Considerations of invariance under the permutation of the indices of the k populations suggest taking random samples of a common size n from each population. Let Y_i be a function of the sufficient statistic (when it exists) for θ_i and let its df be $F_n(\cdot\,; \theta_i)$; this df plays

the role of $F(.; \theta_i)$ of the above discussion. In order that the procedure R of Section 2 satisfy (1.1), the smallest n should be determined such that (2.2) or (2.4) ((2.12) or (2.14)) is at least as large as the preassigned constant γ. Such a solution exists if Y_i's are consistent. As an illustration let π_i be $N(\theta_i, 1)$, i = 1,...,k. Then Y_i's are sample means based on random samples each of size n and $F_n(y, \theta_i) = \Phi(n^{1/2}(y - \theta_i))$ where $\Phi(.)$ is the standard normal df. Now (2.4) gives

$$(4.1) \qquad \inf_{\Omega(\delta)} P(\underset{\sim}{\theta}) = t \int_{-an^{1/2}}^{bn^{1/2}} \Phi^{k-t}(y + n^{1/2}\delta)[1 - \Phi(y)]^{t-1} d\Phi(y) ,$$

where $b \geq a$. The right side of (4.1) tends to unity for $b \geq a > 0$, so that there is a unique n satisfying (1.1).

5. Concluding Remarks.

It should be noted that if $\delta = 0$ ($\rho = 1$) then the integral (2.4) (integral (2.14)) can be evaluated with the help of the incomplete beta function tables and the tables of the df F; in addition if $a = b = \infty$ ($a = \infty$, $b = 0$), $\inf_{\Omega} P(\underset{\sim}{\theta}) = 1/\binom{k}{t}$.

In this formulation of interval estimation and simultaneous selection, the upper confidence bound for $\theta_{[k-t+1]}$ can be obtained by taking $b = \infty (b = 0)$ and some finite a, satisfying conditions of Theorem 1 (Theorem 2). However, the conditions of Theorem 1 (Theorem 2) do not permit the construction of the lower confidence bound for $\theta_{[k-t+1]}$ except in the trivial case $a = \infty$, $b = \infty$ ($a = \infty$, $b = 0$).

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